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OPTIMAL INTERACTION OF INDENTER WITH INHOMOGENEOUS PLATE

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ABSTRACT

Consideration is given to a new class of problems dealing with an optimal design of inhomogeneous plate during dynamic penetration of the rigid indenter. The quality criterion of the process is defined by the specific mass of the target, which absorbs the given kinetic energy of the indenter. Parameters of control are expressed in terms of mechanical characteristics, i.e. distribution of density ρ and the related hardness H across the plate thickness. The maximum principle of Pontryagin are used to search for piece-wise continuous control function. With consideration of impact conditions and characteristics for a given class of material an optimal target structure criterion has been estimated for engineering application.

INTRODUCTION

The problem of searching for mechanical characteristics of inhomogeneous plate subject to impact of a rigid body has been stated first in [1] in the framework of theory of optimal control. This study employs Pontryagin principle of maximum [2] to obtain an optimal structure for a plate with minimal thickness and prescribed specific mass. At present a considerable attention is focused on the problem of structure optimization as applied to the case of inhomogeneous plate of a minimal specific mass using both linear $H(\rho) = A\rho + B$ [3] and nonlinear $H(\rho) = \varphi(\rho)$ [4] relations. An approximate approach to the analysis of penetration process, based on the empirical relation [5] allows to obtain rather simple criteria for structure optimization.

ANALYSIS

1. Formulation of the problem.

According to the applied theory of the plate specific resistance p, penetration of the rigid indenter can be expressed [5] as

$$p = H + k\rho v^2 \tag{1.1}$$

where H is dynamic hardness; k is the shape factor of the indenter head (in case of a tapered head $k=\sin^2\alpha$, α is a half angle of the cone opening); v is the current penetration rate.

The equation of motion for indenter is given as [3]

$$(1/2)M\, d(v^2)/dL = -2\pi \int_0^L p(x,L)\, r(\xi)\partial r(\xi)/\partial\xi \, d\xi, \tag{1.2}$$

where L is the current penetration depth; $r(\xi)$ is the expression for generating line of axisymmetrical indenter; $\xi = L-x$ is the

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coordinate relative to the tip of the indenter (fig. 1).

Distributions of density and hardness are assumed to meet the following requirements

$$\rho \subseteq \Omega,\ \Omega = \{\ \rho(x):\ \rho_1 \le \rho(x) \le \rho_2,\ (x \subseteq [0, L_k])\ \},$$

$$H \subseteq \mathscr{H},\ \mathscr{H} = \{\ H(x):\ H_1 \le H(x) \le H_2,\ (x \subseteq [0, L_k])\ \} \qquad (1.3)$$

We shall restrict ourselves to a class of materials for which there exists a one-to-one mapping φ of the set Ω in the set $\mathscr{H}$: $H = \varphi(\rho)$, $\rho_1 \Rightarrow H_1$, $\rho_2 \Rightarrow H_2$ with $\partial\varphi/\partial\rho > 0$. According to the quality criterion stated below, materials inconsistent with the latter conditions are considered inadequate. Further it is assumed that each of the materials is plastic enough and impact velocities lie within the range, in which application of the relation (1.2) proved to be valid.

The boundary conditions for (1.2) imply that the indenter moves with the initial penetration rate $v(h=0) = v_o$ and reaches some unknown finite penetration depth L_k for which $v(L_k) = 0$. We shall concentrate on the case with the plate thickness being equal to the finite penetration depth $b = L_k$.

Specific mass of such target is taken as the principle criterion of quality

$$J = \min_{\rho \subseteq \Omega} \left\{ \int_0^{L_k} \rho(x)\, dx \right\} \qquad (1.4)$$

2. Conical indenter. Linear relation $H = \varphi(\rho)$.

Let us consider the technique of applying the maximum principle [2] to a number of particular problems. Within the framework of theory of optimal control the problem may be expressed in the form of ($\{y^1 \equiv v^2,\ y^2,\ y^3\}$ is the vector of phase coordinates, $t \equiv L$ is the time analog)

$$dy^1/dt = -E\ [(1/2)B\ t^2 + (A + ky^1)y^2],$$

$$dy^2/dt = y^3,\quad dy^3/dt = \rho, \qquad (2.1)$$

where $E = (4\pi/M)\ (\mathrm{tg}\alpha)^2$.

In the following it is reasonable to introduce additional phase coordinates y^2, y^3, since the right side of the equation of motion involves an explicit form of functional (1.3), leading to condition $d\psi_o/dt = -\partial h/\partial y^o \ne 0$.

The mapping $H = \varphi(\rho)$ is assumed in the form of linear approximation

$$H(\rho) = A\rho(x) + B,$$

$$A = (H_2 - H_1)/(\rho_2 - \rho_1),\quad B = (H_1\rho_2 - H_2\rho_1)/(\rho_2 - \rho_1), \qquad (2.2)$$

At the initial moment of time the vector of phase coordinates remains fixed $t = 0$: $y^1 \equiv y^1_o$, $y^2 = y^3 = 0$. The finite vector value $\bar{y}_k$ belongs to a smooth, two-dimensional variety S_k Euclidean space with dimension $n = 3$

$$S_k\ :\ F_k(y^1_k,\ y^2_k,\ y^3_k) \equiv y^1_k = 0 \qquad (2.3)$$

The condition of transversality for the vector of conjugate variables ψ yields two relations

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$$t = t_k \;:\; \psi_2 = 0,\;\; \psi_3 = 0 \tag{2.4}$$

The equation for the conjugate variables takes the form

$$d\psi_1/dt = Eky^2\psi_1,\;\; d\psi_2/dt = E(A + ky^1)\psi_1,\;\; d\psi_3/dt = -\psi_2 \tag{2.5}$$

Optimization of the process $\rho^o, \bar{y}^o$ requires the existence of such nontrivial constant $\psi_o \le 0$ and vector-function $\psi(t)$ with will allow to meet the maximum condition [2]

$$\max_{\rho \in \Omega} h(\psi(t), \bar{y}(t), t, \rho) = h(\psi(t), \bar{y}(t), t, \rho^o) \tag{2.6}$$

and transversality condition

$$h(\bar{\psi}(t_k), \bar{y}(t_k), t_k, \rho^o(t_k)) = \sum_n \psi_n(t_k) q^n \tag{2.7}$$

where $\bar{q} = \left\{0,\; dy^2/dt,\; dy^3/dt\right\}\Big|_{t=t_k}$.

According to (2.4) the right-hand side of (2.7) is equal to zero.

Hamiltonian operator is expressed as

$$h = (\psi_o + \psi_3)\rho - \psi_1 E\left[(1/2)Bt^2 + (A + ky^1)y^2\right] + \psi_2 y^3 \tag{2.8}$$

Integration of the system (2.5) combined with conditions (2.4), (2.7) enables one to define behavior of Hamiltonian h in terms of linear function of ρ with coefficient $\Phi = \psi_o + \psi_3$.

3. *Cylindrical indenter with a conic head of the height δ. Linear relation $H = \varphi(\rho)$.*

The system of differential equations describing the process is divided into two parts:

$$dy^1/dt = \begin{cases} -E\,[(1/2)B\,t^2 + (A + ky^1)y^2], & t < \delta \\ -E\,[(1/2)B\,\delta^2 + (A + ky^1)y^2], & t \ge \delta, \end{cases} \tag{3.1}$$

$$dy^2/dt = \begin{cases} y^3, & t < \delta \\ y^3 - \delta\rho(t-\delta), & t \ge \delta, \end{cases} \qquad dy^3/dt = \begin{cases} \rho(t), & t < \delta \\ \rho(t) - \rho(t-\delta), & t \ge \delta, \end{cases}$$

Hamiltonian operator takes the form

$$h = (\psi_o + \psi_3)\rho(t) - \psi_1 E\left[(1/2)Bt^2 + (A + ky^1)y^2\right] + \psi_2 y^3, \text{ for } t < \delta;$$

$$\begin{aligned} h = (\psi_o + \psi_3)\rho(t) - (\delta\psi_2 - \psi_3)\rho(t-\delta) + \psi_2 y^3 - \\ - \psi_1 E\left[(1/2)B\delta^2 + (A + ky^1)y^2\right], \text{ for } t \ge \delta \end{aligned} \tag{3.2}$$

The equations for the conjugate variables is expressed as (2.5) $\forall t \in [0, t_k]$. It is assumed that the value y^1_o is such that

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condition $L_k > \delta$ is satisfied automatically.

4. Cylindrical indenter with a flat end-face. Nonlinear relation $H = \varphi(\rho)$.

The equation of motion for indenter and Hamiltonian are written, respectively as

$$dy^1/dt = -E\,[\varphi(\rho) + \rho y^1], \tag{4.1}$$

$$h = \psi_0 \rho - \psi_1 E[\varphi(\rho) + \rho y^1] \tag{4.2}$$

Using differential equation for conjugate variables and transversality condition gives:

$$h = \rho\Phi_1 + \varphi(\rho)\Phi_2 \tag{4.3}$$

$$\Phi_1 = 1 + y^1 fZ, \quad \Phi_2 = fZ, \quad Z = \exp\left(-E\int_t^{t_k} \rho d\tau\right) \tag{4.4}$$

For a piecewise-linear relation $\varphi(\rho)$ (see fig.2) Hamiltonian is transformed to a piecewise-linear function ρ with the slope Φ

$$\Phi = \begin{cases} -1 + (Z\rho_k/(B+A\rho_k))(A+y^1), & \rho\in[\rho_1, \rho_*] \\ -1 + (Z\rho_k/(B_1+D\rho_k))(D+y^1), & \rho\in(\rho_*, \rho_2] \end{cases} \tag{4.5}$$

For nonlinear relation $\varphi(\rho)=B+A\rho^n$ $(A>0, n>0)$ Hamiltonian reduces to

$$h = \rho\Phi(\rho) + B\Phi_2, \quad \Phi = 1 + f_n Z(y^1 + A\rho^{n-1}), \quad f_n = \rho_k/(B+A\rho_k^n) \tag{4.6}$$

The conditions assumed for existence of continuous solutions may be expressed as

$$(\partial h/\partial\rho)\Big|_{\rho=\rho^o} = \Phi_1 + A_n\Phi_2(\rho^o)^{n-1} = 0 \tag{4.7}$$

$$(\partial^2 h/\partial\rho^2)\Big|_{\rho=\rho^o} = A_n n(n-1)\Phi_2(\rho^o)^{n-2} < 0 \tag{4.8}$$

Following (4.7) one gets:

$$\rho^o = \left[-\Phi_1/(A_n\Phi_2)\right]^m, \quad m = 1/(n-1) \tag{4.9}$$

Differentiation of (4.9), using (4.1) gives

$$\partial\rho^o/\partial t = -G(\rho^o)^2\left[1 + B/(A(1-n))\,(\rho^o)^{-n}\right], \quad G = E/n \tag{4.10}$$

RESULTS

Without going into details we shall examine some qualitative results obtained for a number of special cases.

1. Linear relation $H = \varphi(\rho)$.

From the analysis of the system (2.1),(2.5),(2.8) we can draw the following qualitative conclusions :

(i) There is an interval $(t_*, t_k]$ in which the optimal functi-

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on has the form of $\rho^{o} = \rho_{1}$, i.e. the rear layer should be made of more light and less hardness materials;

(ii) The function $\rho^{o} = \rho_{1}$ is optimal within the entire interval $(t_{o}, t_{k}]$, if $B \geq 0$;

(iii) In the case of $B < 0$ the structure of target should be double layer with the front lay being made of hard and heavy material. In this case a relay-type control is realized.

Similar results have been obtained for the case (3.1),(3.2). A general phase diagram of the optimal structure is shown in fig.3.

Here v_{1} is the velocity at which $L_{k} = \delta$, where δ the height of the indenter head. It is seen that the structure largely depends on the parameter B. If the specific hardness $Q = H/\rho$ is assumed the measure of material quality, then, according to (2.2), the condition $B<0$ identifies the maximum quality of heavy material $\{\rho_{2}, H_{2}\}$. In this case a double layer plate is an optimal structure for a target. Contrary, when $v_{1} > v_{o}$ an optimal structure may be represented by a homogeneous plate made from a light material.

2. Nonlinear relation $H = \varphi(\rho)$.

(i) For a piece-wise linear function $H(\rho)$ as plotted in fig.2 the problem is solved for three different materials. The "phase-diagram" of the optimal structure is shown in fig.4, where $\alpha = -(\rho_{1}/\rho_{2})(\rho_{2}-\rho_{*})/(\rho_{*}-\rho_{1})$, $X = (L_{k}/\delta)^{2}-1$. As it follows from the observable scale effect, the optimal structure for a given set of materials (B, B_{1}) depends on the relation L_{k}/δ.

(ii) For nonlinear dependence $H(\rho)$ the function of the optimal control may not include discontinuities. Inequality (4.8) is valid for $n < 1$ in the neighborhood of $(t_{*}, t_{k}]$, $0 < t_{*} < t_{k}$. The procedure of qualitative estimating the type of solution to the equation (4.10) may be as follows. The first approximation (the expression in square brackets in (4.10) is constant) follows from $\partial\rho^{o}/\partial t = -G(\rho^{o})^{2}$ as $(\rho^{o})^{-1} = \rho_{2}^{-1} + G(t-t_{*})$. This solution is found to be exact for $B = 0$. The second approximation has a more complicated form. The results are shown in fig.5. The position (coordinate) of the point t_{*} is calculated numerically and may coincide with the starting point of the process.

The results of present investigation allow to make a prompt qualitative estimation of the optimal target structure. The best ratio of layers in a double-layer target may be calculated numerically by solving the equation of motion for indenter.

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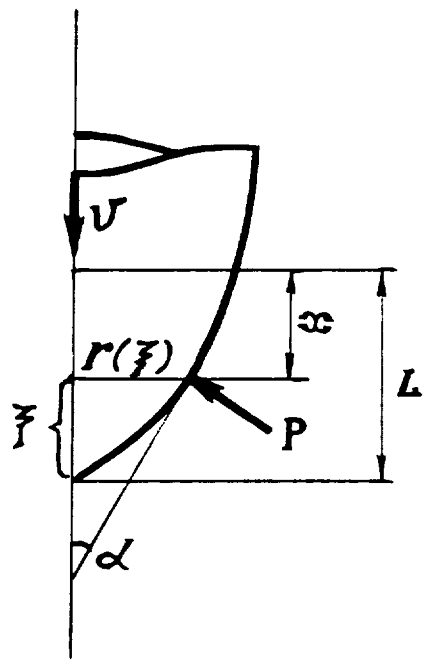

Fig. 1.

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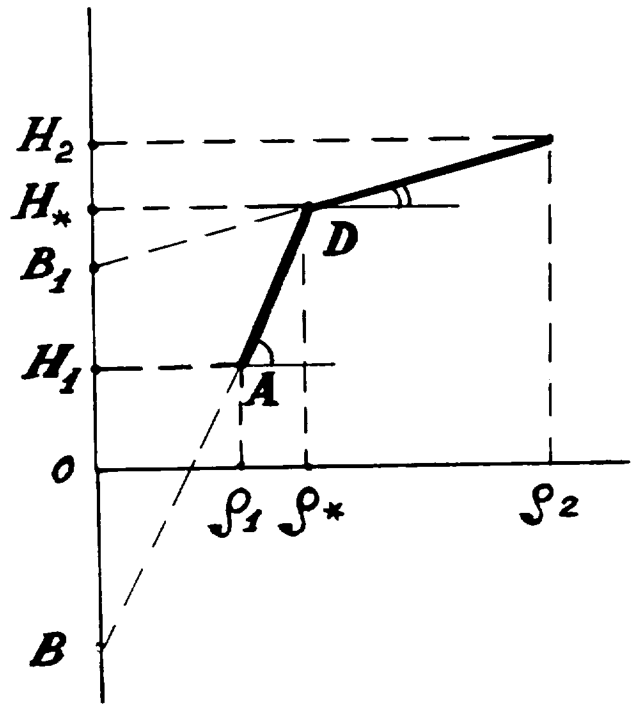

Fig. 2.

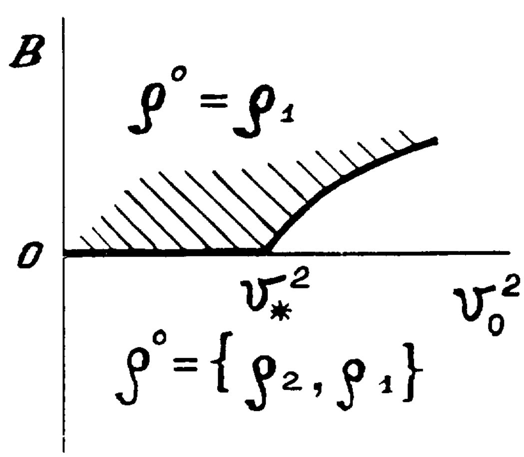

Fig. 3.

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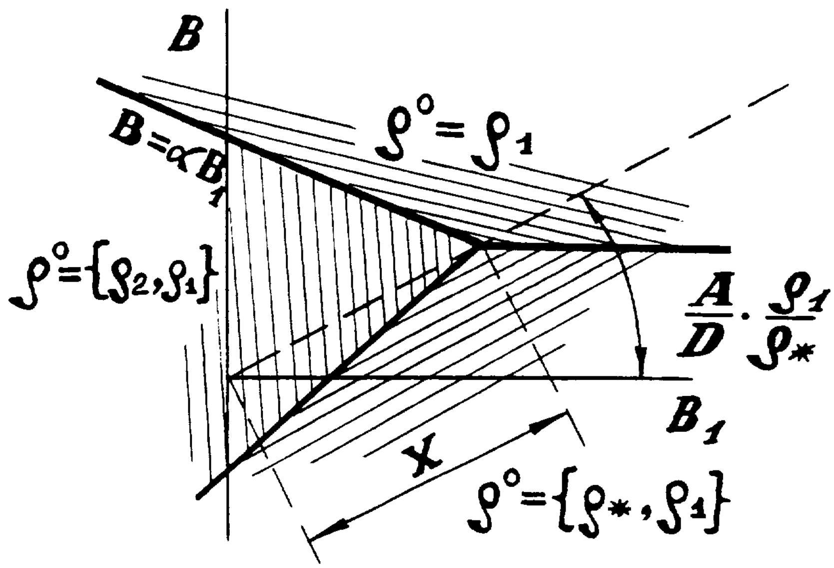

Fig. 4.

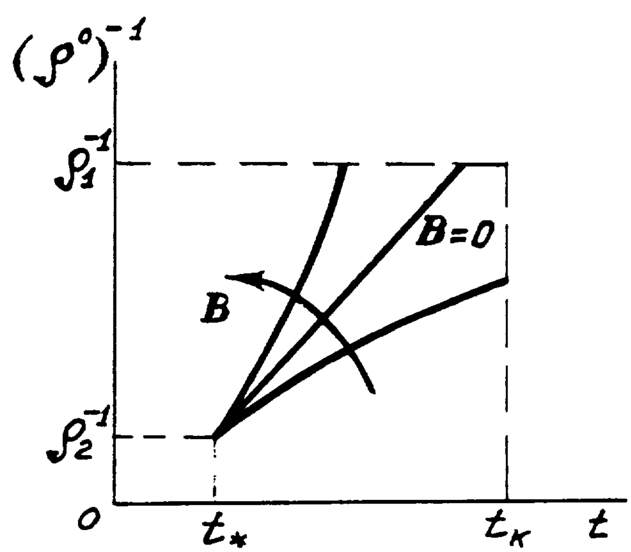

Fig. 5.